Electric Circuits
Discussion 13

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Jun.5th 2018
Contents

• Homework 9
• Review Part 1: Passive Filters
• Review Part 2: Active Filters
• Review Part 3: Laplace Transform
• Review Part 4: Inverse Laplace Transform
Question 8
Question 8
Question 9

[Diagram with a circuit diagram showing a voltage source of 120° V rms, a transformer with 1000 turns and 200 turns, and two impedances: 30 + j12 Ω and 20 - j40 Ω.]
Question 1

Determine the resonant frequency of the circuit shown below.
Given that $R = 1\, \text{k}\Omega$, $L = 10\, \text{mH}$, and $C = 10\, \text{nF}$. 
For the circuit shown below, determine the transform function $H = \frac{V_o}{V_i}$, and determine the frequency $w$ at which $H$ is purely real.
Question 3

(a) \( H(\omega) = \frac{4 \times 10^4(60 + j6\omega)}{(4 + j2\omega)(100 + j2\omega)(400 + j4\omega)} \)
Question 3

(b) \( H(\omega) = \frac{(1 + j0.2\omega)^2(100 + j2\omega)^2}{(j\omega)^3(500 + j\omega)} \)
Question 3

\[(c) \quad H(\omega) = \frac{8 \times 10^{-2}(10 + j10\omega)}{j\omega(16 - \omega^2 + j4\omega)}\]
Question 3

\[ H(\omega) = \frac{4 \times 10^4 \omega^2 (100 - \omega^2 + j50\omega)}{(5 + j5\omega)(200 + j2\omega)^3} \]
Question 4

Fig 4.1
Question 4

Fig 4.2
Question 4
2. Passive Filters
**Type of filters**

- Circuit designed to retain certain frequency range but discard or attenuate others
  - *Low-pass*: pass low frequencies and reject high frequencies
  - *High-pass*: pass high frequencies and reject low frequencies
  - *Band-pass*: pass some particular range of frequencies, reject other frequencies outside that band
  - *Band-reject (notch)*: reject a range of frequencies and pass all other frequencies
Passive Filters

• A filter is passive if it consists only of passive elements
  ▪ R, L, and C.

• LC filters have been used in practical applications for more than eight decades.
  ▪ Very important circuits
    – many technological advances would not have been possible without the development of filters.
  ▪ LC filter technology feeds many areas
    – equalizers, impedance-matching networks, transformers, shaping networks, power dividers, attenuators, and directional couplers …
Passive Filters ---- lowpass filter (RC Circuit)

- A typical lowpass filter is formed when the output of a RC circuit is taken off the capacitor.

\[
H(\omega) = \frac{1}{1 + j\omega RC}
\]

- The -3dB (half power) frequency is:

\[
\omega_c = \frac{1}{RC}
\]

  - Also referred as the cutoff frequency.
  - Filter is designed to pass from DC up to \(\omega_c\).

\[
H(0) = 1, \ H(\infty) = 0.
\]
A typical highpass filter is formed when the output of a RC circuit is taken off the resistor.

\[ H(\omega) = \frac{1}{1 + \frac{1}{j\omega RC}} \]

The cutoff frequency will be the same as the lowpass filter.

\[ \omega_c = \frac{1}{RC} \]

The difference being that the frequencies passed go from \( \omega_c \) to infinity.

\[ H(0) = 0, \quad H(\infty) = 1 \]
Passive Filters ---- lowpass / highpass filter (RL Circuit)

How can you get it easily?
Passive Filters ---- bandpass filter (RLC Circuit)

• A typical bandpass filter is formed when the output of a RLC circuit is taken off the resistor.

\[ H(\omega) = \frac{R}{R + j(\omega L - \frac{1}{\omega C})} \]

• In this case, we will determine the center frequency.

\[ \omega_0 = \sqrt{\frac{1}{LC}}. \]

• The center frequency is the resonant frequency. For bandpass filters, the magnitude of the transfer function is a maximum at the center frequency.
Passive Filters ---- bandstop filter (RLC Circuit)

- A typical bandstop filter is formed when the output of a RLC circuit is taken off the capacitor and.

\[ H(\omega) = \frac{j(\omega L - \frac{1}{\omega C})}{R + j(\omega L - \frac{1}{\omega C})} \]

- In this case, we will determine the frequency of rejection.

\[ \omega_0 = \sqrt{\frac{1}{LC}}. \]

- The frequency of rejection is the **resonant frequency**. For bandstop filters, the magnitude of the transfer function is a **minimum** at the center frequency.
Passive Filters ---- Summary

• Cutoff frequency: the frequency for which the transfer function magnitude is decreased by the factor from its maximum value.

\[ |H(j\omega_c)| = \frac{1}{\sqrt{2}}H_{max}, \]

• Under the cutoff frequency, it is also regarded as the frequency at which the power dissipated in a circuit is half of its maximum value.

• The key to get the type of filter: transfer function! Remind that the above is only the model! You must understand the definition.

• The way to get transfer function: Thevenin / Norton, or use basic equality.
Question 5
Question 5
3. Active Filters
Active Filters

• It is possible, using **op-amps**, together with resistors and capacitors, to create all the common filters.
  • Their ability to isolate input and output also makes them very desirable.
  • Limited to frequency less than 1MHz.
Active Filters ---- lowpass filter (RC Circuit)

• The general first-order active filter:

$$H(\omega) = \frac{V_o}{V_i} = -\frac{Z_f}{Z_i}$$

• Replace the variable, we can get that:

$$H(\omega) = -\frac{R_f}{R_i} \frac{1}{1 + j\omega C_f R_f}$$

• Notice that \( H(0) = -\frac{R_f}{R_i}, H(\infty) = 0 \)

• It is a inverting transformation.

• The corner frequency:

$$\omega_c = \frac{1}{R_f C_f}$$

• This means that several inputs with different \( R_i \) could be summed if required, and the corner frequency would remain the same for each input.
Active Filters ---- highpass filter (RC Circuit)

- The general first-order active filter:

\[ H(\omega) = \frac{V_o}{V_i} = -\frac{Z_f}{Z_i} \]

- Replace the variable, we can get that:

\[ H(\omega) = -\frac{R_f}{R_i + 1/j\omega C_i} = -\frac{j\omega C_i R_f}{1 + j\omega C_i R_i} \]

- Notice that \( H(0) = 0, H(\infty) = -\frac{R_f}{R_i} \)

- It is a inverting transformation.
Active Filters ---- bandpass filter (RC Circuit)

- We can get active bandpass filter by cascading active filters.
- It is a inverting transformation. We can add inverting at last.
- The lowpass section sets the upper corner frequency as
Active Filters ---- bandpass filter (RC Circuit)

Stage 1
Low-pass filter
sets $\omega_2$ value

Stage 2
High-pass filter
sets $\omega_1$ value

Stage 3
An inverter
provides gain

$$H(\omega) = \frac{V_o}{V_i} = \left( -\frac{1}{1 + j\omega C_1 R} \right) \left( -\frac{j\omega C_2 R}{1 + j\omega C_2 R} \right) \left( -\frac{R_f}{R_i} \right)$$
Active Filters ---- bandpass filter (RC Circuit)

- The lowpass section sets the upper corner frequency as

$$\omega_2 = \frac{1}{RC_1}$$

- while the highpass section sets the lower corner frequency as

$$\omega_1 = \frac{1}{RC_2}$$

- The center frequency, bandwidth, and quality factor are found as follows:

$$\omega_0 = \sqrt{\omega_1 \omega_2}$$

$$B = \omega_2 - \omega_1$$

$$Q = \frac{\omega_0}{B}$$

- The pass-band gain (at center frequency by transfer function) is

$$K = \frac{R_f}{R_i} \frac{\omega_2}{\omega_1 + \omega_2}$$
Active Filters ---- bandstop filter (RC Circuit)

- We can get active bandpass filter by cascading active filters.

- It is an inverting transformation.
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Active Filters ---- bandstop filter (RC Circuit)

\[
H(\omega) = \frac{R_f}{R_i} \left( \frac{1}{1 + j\omega/\omega_2} + \frac{j\omega/\omega_1}{1 + j\omega/\omega_1} \right)
\]

\[
= \frac{R_f}{R_i} \frac{(1 + j2\omega/\omega_1 + (j\omega)^2/\omega_1\omega_1)}{(1 + j\omega/\omega_2)(1 + j\omega/\omega_1)}
\]
Active Filters ---- bandstop filter (RC Circuit)

- The lowpass section sets the upper corner frequency as
  \[ \omega_2 = \frac{1}{RC_1} \]

- while the highpass section sets the lower corner frequency as
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- The center frequency, bandwidth, and quality factor are found as follows:
  \[ \omega_0 = \sqrt{\omega_1 \omega_2} \]
  \[ B = \omega_2 - \omega_1 \]
  \[ Q = \frac{\omega_0}{B} \]

- The gain between 2 passbands (by transfer function at frequency is 0 or infinity) is
  \[ K = \frac{R_f}{R_i} \]
The difference between Passive Filters and Active Filters

- The maximum magnitude of transfer function does not exceed 1 in passive filters. Active filters provide a control over amplification not available in passive filter circuits.

- Recall that both the cutoff frequency and the passband magnitude of passive filters were altered with the addition of a resistive load at the output of the filter. This is not the case with active filters, due to the properties of op amps. Thus, we use active circuits to implement filter designs when gain, load variation, and physical size are important parameters in the design specifications.
Question 6
Question 7
3. Laplace Transform
Why we need transforms?

- A mathematical conversion from one way of thinking to another to make a problem easier to solve.
Laplace Transforms

- $t$ is real, $s$ is complex!
- Assumes $f(t) = 0$ for all $t<0$ (Single-sided Laplace transformation)
- Note “transform”: $f(t) \rightarrow F(s)$, where $t$ is integrated and $s$ is variable.
- Conversely, $F(s) \rightarrow f(t)$, $t$ is variable and $s$ is integrated.

$$F(s) = L\{f(t)\} = \int_{0}^{\infty} f(t)e^{-st} dt$$

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds$$

$$s = \sigma + j\omega$$
Laplace Transforms

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$$s = \sigma + j\omega$$
Laplace Transforms -- example

\[ \frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} u(t) \]

\[ v(0^+) = v_0 \]

\[ \frac{d^2v}{dt^2} + \frac{R}{L} \frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC} \]

\[ v(0^+) = v_0 \quad \frac{dv(0^+)}{dt} = \ldots \]
Laplace Transforms -- example

We can use Laplace transforms to turn an initial value problem (IVP)

\[ y'' + 3y' - 4y = t \cdot u(t-1) \]
\[ y(0) = -1, \quad y'(0) = 2 \]

into an algebraic problem

\[ Y(s) \ast (s^2 + 3s - 4) + (s + 1) = \frac{s + 1}{s^2 \cdot e^s} \]

Solve for \( y(t) \)

Solve for \( Y(s) \)
Laplace Transforms -- example

If you solve the algebraic equation

\[ Y(s) = \frac{-(s + 1) \cdot (s^2 \cdot e^s - 1) \cdot e^{-s}}{s^2 \cdot (s^2 + 3s - 4)} \]

and find the inverse Laplace transform of the solution, \( Y(s) \), you have the solution to the IVP.
Laplace Transforms -- example

The inverse Laplace transform of

\[ Y(s) = \frac{-(s + 1) \cdot (s^2 \cdot e^s - 1) \cdot e^{-s}}{s^2 \cdot (s^2 + 3s - 4)} \]

is

\[ y(t) = u(t - 1)\left(\frac{2}{5e} \cdot e^t + \frac{3e^4}{80} \cdot (e^t)^{-4} - \frac{1}{4} t - \frac{3}{16}\right) \]

\[ -u(t)\left(\frac{2}{5} \cdot e^t - \frac{3}{5} \cdot (e^t)^{-4}\right) \]
Laplace Transforms -- example

Thus

\[ y(t) = u(t - 1) \left( \frac{2}{5e} \cdot e^t + \frac{3e^4}{80} \cdot (e^t)^{-4} - \frac{1}{4} t - \frac{3}{16} \right) \]

\[ -u(t) \left( \frac{2}{5} \cdot e^t - \frac{3}{5} \cdot (e^t)^{-4} \right) \]

is the solution to the I.V.P.

\[ y'' + 3y' - 4y = t \cdot u(t - 1) \]

\[ y(0) = -1, \; y'(0) = 2 \]
Laplace Transforms

- **Time domain**
  - Linear differential equation
  - Laplace transform
  - Algebra
  - Laplace transformed equation

- **Laplace domain or complex frequency domain**
  - Laplace solution

- **Time domain**
  - Time domain solution
  - Inverse Laplace transform
Existence of Laplace Transforms

\[ F(s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st} \, dt = \int_0^\infty f(t)e^{-\sigma t}e^{-j\omega t} \, dt \quad s = \sigma + j\omega \]

- A sufficient condition: for some real value \( \sigma = \sigma_c \),

\[ \int_0^\infty e^{-\sigma t} |f(t)| \, dt < \infty \]

**Region of convergence (ROC):**

\[ \text{Re}(S) = \sigma > \sigma_c \]
Uniqueness of Laplace Transforms

- There may be different functions whose Laplace transforms are the same.

  \[ f(t) = e^{-t} \quad \text{and} \quad g(t) = \begin{cases} \ e^{-t} & t \neq 3 \\ \ 0 & t = 3 \end{cases} \]

  have the same Laplace transform.

- Lerch’s Theorem:

  \[ f, g : \text{continuous on} \ [0, \infty] \]

  If \( L[f] = L[g] \), then \( f = g \)
Properties of Laplace Transforms

- Linearity
- Time domain
  - Scaling
  - Time shift
  - Integration
  - Differentiation
- Frequency domain
  - “frequency” (or s-plane) shift
  - Frequency differentiation: multiplication by $t^n$
# Properties of Laplace Transforms

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</tr>
<tr>
<td></td>
<td>$\frac{f(t)}{t}$</td>
<td>$\int_s^\infty F(s)ds$</td>
</tr>
<tr>
<td><strong>Time periodicity</strong></td>
<td>$f(t) = f(t + nT)$</td>
<td>$\frac{F_1(s)}{1 - e^{-sT}}$</td>
</tr>
<tr>
<td><strong>Initial value</strong></td>
<td>$f(0)$</td>
<td>$\lim_{s \to \infty} sF(s)$</td>
</tr>
<tr>
<td><strong>Final value</strong></td>
<td>$f(\infty)$</td>
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</tr>
<tr>
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<td>$f_1(t) * f_2(t)$</td>
<td>$F_1(s)F_2(s)$</td>
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</table>
Properties of Laplace Transforms

<table>
<thead>
<tr>
<th>Property</th>
<th>$f(t)$</th>
<th>$F(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$a_1 f_1(t) + a_2 f_2(t)$</td>
<td>$a_1 F_1(s) + a_2 F_2(s)$</td>
</tr>
<tr>
<td>Scaling</td>
<td>$f(at)$</td>
<td>$\frac{1}{a} F\left(\frac{s}{a}\right)$</td>
</tr>
<tr>
<td>Time shift</td>
<td>$f(t - a)u(t - a)$</td>
<td>$e^{-as} F(s)$</td>
</tr>
<tr>
<td>Frequency shift</td>
<td>$e^{-at} f(t)$</td>
<td>$F(s) + a$</td>
</tr>
<tr>
<td>Time differentiation</td>
<td>$\frac{df}{dt}$</td>
<td>$sF(s) - f(0^-)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{d^2f}{dt^2}$</td>
<td>$s^2F(s) - sf(0^-) - f'(0^-)$</td>
</tr>
<tr>
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</tr>
<tr>
<td></td>
<td>$\frac{d^nf}{dt^n}$</td>
<td>$s^nF(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \ldots - f^{(n-1)}(0^-)$</td>
</tr>
<tr>
<td>Time integration</td>
<td>$\int_0^t f(x) dx$</td>
<td>$\frac{1}{s} F(s) $</td>
</tr>
<tr>
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<td>$s^2F(s) - sf(0^-) - f'(0^-)$</td>
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<td>$s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$</td>
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**Laplace Transforms Pairs**

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s)$</th>
<th>$f(t)$</th>
<th>$F(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(t)$</td>
<td>$\frac{1}{s}$</td>
<td>$\sin\omega t$</td>
<td>$\frac{s}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>$u(t)$</td>
<td>$\frac{1}{s}$</td>
<td>$\cos\omega t$</td>
<td>$\frac{s}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>$e^{-at}$</td>
<td>$\frac{1}{s + a}$</td>
<td>$\sin(\omega t + \theta)$</td>
<td>$\frac{s \sin \theta + \omega \cos \theta}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>$t$</td>
<td>$\frac{1}{s^2}$</td>
<td>$\cos(\omega t + \theta)$</td>
<td>$\frac{s \cos \theta - \omega \sin \theta}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
<td>$e^{-at} \sin \omega t$</td>
<td>$\frac{\omega}{(s + a)^2 + \omega^2}$</td>
</tr>
<tr>
<td>$te^{-at}$</td>
<td>$\frac{1}{(s + a)^2}$</td>
<td>$e^{-at} \cos \omega t$</td>
<td>$\frac{s + a}{(s + a)^2 + \omega^2}$</td>
</tr>
<tr>
<td>$t^n e^{-at}$</td>
<td>$\frac{n!}{(s + a)^{n+1}}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Defined for $t \geq 0$; $f(t) = 0$, for $t < 0$.  

Determine the Laplace transform of $f(t) = t^2 \sin 2t \ u(t)$.

**Solution:**
We know that

$$\mathcal{L}[\sin 2t] = \frac{2}{s^2 + 2^2}$$

Using frequency differentiation in Eq. (15.34),

$$F(s) = \mathcal{L}[t^2 \sin 2t] = (-1)^2 \frac{d^2}{ds^2} \left( \frac{2}{s^2 + 4} \right)$$

$$= \frac{d}{ds} \left( \frac{-4s}{(s^2 + 4)^2} \right) = \frac{12s^2 - 16}{(s^2 + 4)^3}$$
Find the Laplace transform of the gate function in Fig. 15.5.

**Solution:**

We can express the gate function in Fig. 15.5 as

\[ g(t) = 10[u(t - 2) - u(t - 3)] \]

Since we know the Laplace transform of \( u(t) \), we apply the time-shift property and obtain

\[ G(s) = 10 \left( \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} \right) = \frac{10}{s} (e^{-2s} - e^{-3s}) \]
Laplace Transforms Example 3

Calculate the Laplace transform of the periodic function in Fig. 15.7.

Solution:
The period of the function is \( T = 2 \). To apply Eq. (15.40), we first obtain the transform of the first period of the function.

\[
f_1(t) = 2t[u(t) - u(t - 1)] = 2tu(t) - 2tu(t - 1)
= 2tu(t) - 2(t - 1 + 1)u(t - 1)
= 2tu(t) - 2(t - 1)u(t - 1) - 2u(t - 1)
\]

Using the time-shift property,

\[
F_1(s) = \frac{2}{s^2} - \frac{2e^{-s}}{s^2} - \frac{2}{s}e^{-s} = \frac{2}{s^2}(1 - e^{-s} - se^{-s})
\]

Thus, the transform of the periodic function in Fig. 15.7 is

\[
F(s) = \frac{F_1(s)}{1 - e^{-Ts}} = \frac{2}{s^2(1 - e^{-2s})}(1 - e^{-s} - se^{-s})
\]

\[
f(t)
\]

\[
\text{2}
\]

\[
0 1 2 3 4 5 t
\]
Laplace Transforms Example 4

Find the initial and final values of the function whose Laplace transform is

\[ H(s) = \frac{20}{(s + 3)(s^2 + 8s + 25)} \]

Solution:
Applying the initial-value theorem,

\[
\begin{align*}
    h(0) &= \lim_{s \to \infty} sH(s) = \lim_{s \to \infty} \frac{20s}{(s + 3)(s^2 + 8s + 25)} \\
    &= \lim_{s \to \infty} \frac{20/s^2}{(1 + 3/s)(1 + 8/s + 25/s^2)} = \frac{0}{(1 + 0)(1 + 0 + 0)} = 0
\end{align*}
\]

To be sure that the final-value theorem is applicable, we check where the poles of \( H(s) \) are located. The poles of \( H(s) \) are \( s = -3, -4 \pm j3 \), which all have negative real parts: They are all located on the left half of the \( s \) plane (Fig. 15.9). Hence, the final-value theorem applies and

\[
\begin{align*}
    h(\infty) &= \lim_{s \to 0} sH(s) = \lim_{s \to 0} \frac{20s}{(s + 3)(s^2 + 8s + 25)} \\
    &= \frac{0}{(0 + 3)(0 + 0 + 25)} = 0
\end{align*}
\]

Figure 15.9
For Example 15.7: Poles of \( H(s) \).
4. Inverse Laplace Transform
(Provided by Prof. Zhou)
Inverse Laplace Transforms

\[ f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds \]

- The formal way to compute \( f(t) \) from \( F(s) \), i.e., the inverse
- We generally do not use this to take the inverse Laplace in Engineering.
  - To use it requires a background in the use of complex variables (complex analysis) and the theory of residues.
  - Fortunately, we can accomplish the same goal (that of taking the inverse Laplace) by using partial fraction expansion and recognizing transform pairs (look up table method).
Look-Up Table Method

- If \( X(s) \) can be written as a **sum of terms** with known Inverse Laplace Transforms, \( x(t) \) will be the sum of these Inverse Laplace Transforms

\[
F(s) = F_1(s) + F_2(s) + \cdots + F_n(s)
\]

\[
f(t) = f_1(t) + f_2(t) + \cdots + f_n(t)
\]

- Requires knowledge or reference of Laplace Transform pairs, but is much simpler than directly calculating the Inverse Laplace Transform
Simple Example

Find the inverse of

\[
F(s) = \frac{10}{s} + \frac{4}{s + 3}
\]

\[
F(s) = \frac{10}{s} + \frac{4}{s + 3}
\]

\[
f(t) = 10u(t) + 4e^{-3t}u(t)
\]
Inverse Laplace Transform - General Form

- In general $F(s)$ can be written as a rational function

$$F(s) = \frac{b_m(s - z_1)(s - z_2)\ldots(s - z_m)}{a_n(s - p_1)(s - p_2)\ldots(s - p_n)}$$

- $z_1, z_2, \ldots, z_m$ are the zeros of $F(s)$
- $p_1, p_2, \ldots, p_n$ are the poles of $F(s)$

- If $F(s)$ is written as a strictly rational function, a method using partial-fraction expansion can be used to determine the inverse Laplace transform.

https://en.wikipedia.org/wiki/Rational_function
Strictly Rational Function

• A function $F(s)$ is strictly rational if the **Degree** of its **Numerator Polynomial** $N(s)$ is **Less** than the **Degree** of its **Denominator Polynomial** $D(s)$.

• If $N(s) \geq D(s)$, perform **Long Division** until the remainder polynomial $R(s)$ is of lesser order than $D(s)$

$$F(s) = \frac{N(s)}{D(s)} = Q(s) + \frac{R(s)}{D(s)}$$

- $N(s) = \text{Numerator}$
- $D(s) = \text{Denominator}$
- $Q(s) = \text{Quotient}$
- $R(s) = \text{Remainder}$
Partial-Fraction Expansion based on Poles

- Once \( F(s) \) is written in a strictly rational form, **Partial Fraction Expansion** can be performed.

- Partial-Fraction Expansion of \( F(s) \) can be classified into **two categories** based on the **Poles** of \( F(s) \)
  - CASE I: All poles are **Distinct**
  - CASE II: All or some poles are **Repeated**
CASE1: Distinct Poles

- Assume that all Poles are Different \((p_i \neq p_j \text{ if } i \neq j)\)

\[
F(s) = \frac{b_m (s - z_1)(s - z_2)\ldots(s - z_m)}{a_n (s - p_1)(s - p_2)\ldots(s - p_n)}
\]

- Assume numerator order is less than denominator order, then the Partial Fraction Expansion of \(F(s)\) is given by:

\[
F(s) = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \ldots + \frac{c_n}{s - p_n}
\]

where:

\[
c_i = \left(s - p_i\right)F(s) \bigg|_{s=p_i}, \, i = 1, 2, \ldots, n
\]

are residues of \(F(s)\).
L^{-1} of Distinct Terms

- F(s) can be written as the sum of terms
  - Due to the **Linearity Property**, f(t) will be the sum of the Inverse Laplace Transform of the terms

\[
F(s) = \sum_{i=1}^{n} \left\{ \frac{C_i}{s - p_i} \right\} \xrightarrow{L} \sum_{i=1}^{n} L^{-1} \left\{ \frac{C_i}{s - p_i} \right\} = f(t)
\]

- Building from previous work, f(t) is the **summation of unit steps multiplied by exponentials**

\[
f(t) = \sum_{i=1}^{n} c_i e^{p_i t} u(t)
\]
Partial-Fraction Expansion Method for Determining Inverse Laplace Transform

1. Put **Rational Function** into **Strictly Rational Form**
   where the degree of the numerator polynomial less than that of the denominator polynomial

2. Factor the **Denominator** Polynomial

3. Perform **Partial-Fraction Expansion**

4. Use **Laplace Transform Pair Table** to obtain the inverse Laplace transform
Example

- Given:

\[ F(s) = \frac{2s^2 - s + 5}{(s + 1)(s + 2)(s - 3)} \]

Use the Partial-Fraction Expansion and the Table Look-Up Method determine the Inverse Laplace Transform of \( F(s) \).
Partial Fraction Expansion

- Start by finding Partial Fraction Expansion of \( F(s) \)

- Poles of \( p_1=-1, p_2=-2, p_3=-3 \)

- Find \( c_i \) coefficients

\[
F(s) = \frac{2s^2 - s + 5}{(s + 1)(s + 2)(s - 3)}
\]

\[
F(s) = \frac{c_1}{s + 1} + \frac{c_2}{s + 2} + \frac{c_3}{s - 3}
\]
Find Coefficient $C_1$

- Each coefficient is determined by evaluating:

$$c_i = (s - p_i)F(s)\bigg|_{s=p_i}, \quad i = 1,2,...,n$$

- $c_1$ is explicitly evaluated as:

$$c_1 = F(s)(s + 1)\bigg|_{s=-1} = \left. \left( \frac{2s^2 - s + 5}{(s + 2)(s - 3)} \right) \right|_{s=-1} = \left. \frac{2(-1)^2 - (-1) + 5}{(-1 + 2)(-1 - 3)} \right|_{s=-1} = -2$$

$$C_1 = -2$$
Find Coefficient $C_2$ and $C_3$


c_2 = F(s)(s + 2) \bigg|_{s=-2} \\
\left. c_2 = \left( \frac{2s^2 - s + 5}{(s + 1)(s + 2)(s - 3)} \right)(s + 2) \right|_{s=-2}
\Rightarrow c_2 = 3

\[ c_2 = \frac{2(-2)^2 - (-2) + 5}{(-2 + 1)(-2 - 3)} = 3 \]

\[ c_3 = F(s)(s - 3) \bigg|_{s=3} \\
\left. c_3 = \left( \frac{2s^2 - s + 5}{(s + 1)(s + 2)(s - 3)} \right)(s - 3) \right|_{s=3}
\Rightarrow c_3 = 1

\[ c_3 = \frac{2(3)^2 - (3) + 5}{(3 + 1)(3 + 2)} = 1 \]
Partial Fraction Expanded $F(s)$

- **Original Expression**

$$F(s) = \frac{2s^2 - s + 5}{(s + 1)(s + 2)(s - 3)}$$

- **Expansion**

$$F(s) = \frac{c_1}{s + 1} + \frac{c_2}{s + 2} + \frac{c_3}{s - 3}$$

- **Replace with Coefficients**

$$F(s) = \frac{-2}{s + 1} + \frac{3}{s + 2} + \frac{1}{s - 3}$$
Inverse Laplace Transform

\[ f(t) = \text{Sum of the Inverse Laplace Transform of the individual terms of } F(s) \]

\[ F(s) = \frac{-2}{s + 1} + \frac{3}{s + 2} + \frac{1}{s - 3} \]

\[ f(t) = -2e^{-t}u(t) + 3e^{-2t}u(t) + e^{3t}u(t) \]
Exercise

• Find the inverse Laplace transform of

\[ F(s) = \frac{12(s + 1)(s + 3)}{s(s + 2)(s + 4)(s + 5)} \]

\[ f(t) = \left( \frac{9}{10} + e^{-2t} + \frac{36}{8} e^{-4t} - \frac{32}{5} e^{-5t} \right) u(t) \]
Special Case: Complex Poles

- Find the inverse Laplace Transform of

$$Y(s) = \frac{10(s + 2)}{s(s^2 + 4s + 5)}$$

$$Y(s) = \frac{10(s + 2)}{s(s + 2 - j1)(s + 2 + j1)} = \frac{K_0}{s} + \frac{K_1}{s + 2 - j1} + \frac{K_2}{s + 2 + j1}$$

$$K_0 = sY(s)|_{s=0} = \frac{10(2)}{(2 - j1)(2 + j1)} = \frac{20}{5} = 4$$

$$K_1 = (s + 2 - j1)Y(s)|_{s=-2+j1} = \frac{10(j1)}{(-2 + j1)(j1)} = \frac{5}{\sqrt{5} \angle 153.43^\circ} = 2.236 \angle -153.43^\circ = 2.236e^{-j2.678}$$

$$K_2 = (s + 2 + j1)Y(s)|_{s=-2-j1} = \frac{10(-j1)}{(-2 - j1)(-j1)} = \frac{5}{\sqrt{5} \angle -153.43^\circ} = 2.236 \angle 153.43^\circ = 2.236e^{j2.678}$$

$$K_1 = K_2^*$$

MUST use radians in exponent
\[ Y(s) = \frac{10(s + 2)}{s(s + 2 - j1)(s + 2 + j1)} = \frac{4}{s} + \frac{2.236e^{-j2.678}}{s + 2 - j1} + \frac{2.236e^{j2.678}}{s + 2 + j1} \]

\[ y(t) = (4 + 2.236e^{-j2.678} \cdot e^{-(2-j1)t} + 2.236e^{j2.678} \cdot e^{-(2+j1)t})u(t) \]

\[ y(t) = \left(4 + 2.236e^{-2t} \left(e^{j(t-2.678)} + e^{-j(t-2.678)}\right)\right)u(t) \]

\[ y(t) = \left(4 + 2 \times 2.236e^{-2t} \cos(t - 2.678)\right)u(t) \]
General Result

Whenever $D(s)$ contains distinct complex roots — that is, factor of $(s + \alpha - j\beta)(s + \alpha + j\beta)$, then a pair of terms of the form

$$\frac{K}{s + \alpha - j\beta} + \frac{K^*}{s + \alpha + j\beta}$$

appears in the partial fraction expansion, where the coefficient, $K$, is in general a complex number.

$$K = |K|e^{j\theta} = |K|\angle\theta \quad K^* = |K|e^{-j\theta} = |K|\angle - \theta$$

$$L^{-1}\left\{\frac{K}{s + \alpha - j\beta} + \frac{K^*}{s + \alpha + j\beta}\right\} = 2|K|e^{-\alpha t} \cos(\beta t + \theta)$$
Exercise

- Find the inverse Laplace transform of

\[ F(s) = \frac{100(s + 3)}{(s + 6)(s^2 + 6s + 25)} \]

\[ [-12e^{-6t} + 20e^{-3t} \cos(4t - 53.13^\circ)]u(t) \]
Another Way

\[ Y(s) = \frac{10(s + 2)}{s(s^2 + 4s + 5)} \]

Using quadratic factors

\[ Y(s) = \frac{10(s + 2)}{s(s^2 + 4s + 5)} = \frac{C_0}{s} + \frac{C_1(s + 2)}{(s + 2)^2 + 1} + \frac{C_2}{(s + 2)^2 + 1} \]

\[ = \frac{C_0((s + 2)^2 + 1) + C_1(s + 2)s + C_2s}{s(s^2 + 4s + 5)} \]

\[ y(t) = (C_0 + C_1 e^{-2t} \cos t + C_2 e^{-2t} \sin t)u(t) \]

\[ C_0 = 4 \quad C_1 = -4 \quad C_2 = 2 \]
Complex Poles – General Case

- \( F(s) \) may have the general form:

\[
F(s) = \frac{A_1 s + A_2}{s^2 + as + b} + F_1(s)
\]

- Where \( F_1(s) \) is the remaining part that does not have this pair.
- If we complete the square by letting:

\[
s^2 + as + b = s^2 + 2\alpha s + \alpha^2 + \beta^2 = (s + \alpha)^2 + \beta^2
\]

- and we let

\[
A_1 s + A_2 = A_1(s + \alpha) + B_1 \beta
\]

Then \( F(s) \) becomes:

\[
F(s) = \frac{A_1(s + \alpha)}{(s + \alpha)^2 + \beta^2} + \frac{B_1 \beta}{(s + \alpha)^2 + \beta^2} + F_1(s)
\]

\[
f(t) = (A_1 e^{-\alpha t} \cos \beta t + B_1 e^{-\alpha t} \sin \beta t) u(t) + f_1(t)
\]
CASE2: Repeated Pole

- Assume that some or all poles (roots) are repeated
- For the case below, pole $p_1$ is repeated $r$ times

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s - p_1)^r (s - p_{r+1}) \cdots (s - p_n)}$$

- Partial Fraction Expansion shows Repeated and Distinct poles

$$F(s) = \frac{c_1}{s - p_1} + \frac{c_2}{(s - p_1)^2} + \cdots + \frac{c_r}{(s - p_1)^r} + \frac{c_{r+1}}{(s - p_{r+1})} + \cdots + \frac{c_n}{s - p_n}$$

Repeated pole

Distinct poles
Determine Coefficients

- Find **Distinct Poles Coefficients**, \( i = r+1, r+2, \ldots, n \)
  \[
  c_i = (s - p_i)F(s) \bigg|_{s=p_i}, i = r + 1, r + 2, \ldots, n
  \]

- Find **first repeated pole coefficient**, \( i = r \)
  \[
  c_r = (s - p_r)^r F(s) \bigg|_{s=p_i}
  \]

- Find **general repeated poles**, \( i = 1, 2, \ldots, r-1 \)
  \[
  c_i = \frac{1}{(r-i)!} \frac{d^{(r-i)}}{ds^{(r-i)}} \left\{(s - p_1)^r F(s)\right\} \bigg|_{s=p_1}
  \]
Example

- Given:

\[ F(s) = \frac{2s^4 + s^3 - 2s}{(s + 2)^3(s + 1)} \]

- Order of Numerator (4) = Order of Denominator (4) therefore \( F(s) \) is NOT Strictly Rational
- Notice Pole \( s = -2 \) is Repeated 3 times
- Use the Partial-Fraction Expansion and the Table Look-Up Method determine the Inverse Laplace Transform of \( F(s) \)
Solution Steps

• Convert $F(s)$ to a **strictly rational function**
  ▪ Perform **long division**

$$F(s) = \frac{N(s)}{D(s)} = Q(s) + \frac{R(s)}{D(s)}$$

▪ Perform **partial fraction expansion** on rational part of $F(s)$
  ◆ **Calculate coefficients**

$$F(s) = \frac{c_1}{s - p_1} + \frac{c_2}{(s - p_1)^2} + \ldots + \frac{c_r}{(s - p_1)^r} + \frac{c_{r+1}}{(s - p_{r+1})} + \ldots + \frac{c_n}{s - p_n}$$

▪ Use **table method** to determine Inverse Laplace Transform
Repeated Pole Example

- F(s) must be decomposed into a constant plus a Strictly Rational Function

\[ F(s) = \frac{2s^4 + s^3 - 2s}{(s + 2)^3(s + 1)} = Q(s) + \frac{R(s)}{(s + 2)^3(s + 1)} \]

- Q(s) will just be a constant since the order of the numerator and the denominator are the same

\[ c_o = \lim_{s \to \infty} F(s) = 2 \]

\[ F(s) = 2 + \frac{-13s^3 - 36s^2 - 42s - 16}{(s + 2)^3(s + 1)} \]
Expand Rational Part of X(s)

- The rational part of F(s) will be referred to as $F_o(s)$

$$F(s) = 2 + \frac{-13s^3 - 36s^2 - 42s - 16}{(s + 2)^3(s + 1)} = 2 + F_o(s)$$

- Partial Fraction Expansion must be performed on $F_o(s)$

- $F_o(s)$ has 3 repeated roots and one distinct root

$$F_o(s) = \frac{-13s^3 - 36s^2 - 42s - 16}{(s + 2)^3(s + 1)}$$

Partial Fraction Expansion

$$F_o(s) = \frac{c_1}{s + 2} + \frac{c_2}{(s + 2)^2} + \frac{c_3}{(s + 2)^3} + \frac{c_4}{s + 1}$$
Evaluating Coefficient $C_4$

- $C_4$ is evaluated using the distinct pole expression as shown in the previous example

$$c_4 = (s + 1) F_o (s) \bigg|_{s=-1}$$

$$c_4 = (s + 1) \left( \frac{-13 s^3 - 36 s^2 - 42 s - 16}{(s + 1)(s + 2)^3} \right) \bigg|_{s=-1}$$

$$= \left( \frac{-13 s^3 - 36 s^2 - 42 s - 16}{(s + 2)^3} \right) \bigg|_{s=-1}$$

$$= \frac{-13(-1)^3 - 36(-1)^2 - 42(-1) - 16}{(-1+2)^3}$$

$$c_4 = 3$$

$$F_o (s) = \frac{c_1}{s+2} + \frac{c_2}{(s+2)^2} + \frac{c_3}{(s+2)^3} + \frac{c_4}{s+1}$$

$$F_o (s) = \frac{-13 s^3 - 36 s^2 - 42 s - 16}{(s + 2)^3 (s + 1)}$$
Evaluating Coefficient $C_3$

- $C_3$ is evaluated similar to $C_4$

\[ c_3 = (s + 2)^3 F_o(s) \bigg|_{s=-2} \]

\[ = (s + 2)^3 \left( \frac{-13s^3 - 36s^2 - 42s - 16}{(s + 1)(s + 2)^3} \right) \bigg|_{s=-2} \]

\[ = -13(-2)^3 - 36(-2)^2 - 42(-2) - 16 \]

\[ = \frac{(-1 + 2)^3}{(-1 + 2)^3} = -28 \]

\[ c_3 = -28 \]
Repeated Pole Coefficients

- To find $C_2$ and $C_1$, the following expression must be evaluated for each case ($r = 3, p_1 = -2, i = 2, 1$)

$$c_i = \frac{1}{(r-i)!} \frac{d^{(r-i)}}{ds^{(r-i)}} \left\{ (s - p_1)^r F(s) \right\} \bigg|_{s=p_1}$$

$$= \frac{1}{(3-i)!} \frac{d^{(3-i)}}{ds^{(3-i)}} \left\{ (s + 2)^3 \left( \frac{-13s^3 - 36s^2 - 42s - 16}{(s + 1)(s + 2)^3} \right) \right\} \bigg|_{s=-2}$$

$$= \frac{1}{(3-i)!} \frac{d^{(3-i)}}{ds^{(3-i)}} \left\{ \frac{-13s^3 - 36s^2 - 42s - 16}{(s + 1)} \right\} \bigg|_{s=-2}$$

- For simplicity, let $Y(s)$ be the expression to be differentiated

$$Y(s) = \frac{-13s^3 - 36s^2 - 42s - 16}{(s + 1)}$$
Repeated Pole Coefficients

- $C_2$ equals the first derivative of $Y(s)$

$$C_2 = \frac{1}{(3-2)!} \left. \frac{d^{(3-2)}}{ds^{(3-2)}} Y(s) \right|_{s=-2} = \frac{1}{1!} \left. \frac{d}{ds} Y(s) \right|_{s=-2} = Y'(s)$$

- Similarly, $C_1$ equals the second derivative of $Y(s) / 2$

$$C_1 = \frac{1}{(3-1)!} \left. \frac{d^{(3-1)}}{ds^{(3-1)}} \frac{1}{2} Y(s) \right|_{s=-2} = \frac{1}{2!} \left. \frac{d^2}{ds^2} Y(s) \right|_{s=-2} = \frac{1}{2} \left. \frac{d}{ds} Y'(s) \right|_{s=-2}$$

$$C_1 = \frac{Y''(s)}{2}$$

- Before differentiating, $Y(s)$ can be rewritten as:

$$Y(s) = \frac{-13s^3 - 36s^2 - 42s - 16}{(s + 1)} = (-13s^3 - 36s^2 - 42s - 16)(s + 1)^{-1}$$
Calculating First Derivative

- Calculating First Derivative

\[ Y(s) = (-13s^3 - 36s^2 - 42s - 16)(s + 1)^{-1} \]

\[ Y'(s) = (-13s^3 - 36s^2 - 42s - 16)(-1)(s + 1)^{-2} + (s + 1)^{-1}(-39s^2 - 72s - 42) \]

\[ Y'(s) = \frac{-39s^2 - 72s - 42}{(s + 1)} - \frac{-13s^3 - 36s^2 - 42s - 16}{(s + 1)^2} \]

\[ = \frac{(-39s^2 - 72s - 42)(s + 1)}{(s + 1)(s + 1)} - \frac{-13s^3 - 36s^2 - 42s - 16}{(s + 1)^2} \]

\[ = -\left(\frac{26s^3 + 75s^2 + 72s + 26}{(s + 1)^2}\right) \]

\[ c_2 = Y'(s)\bigg|_{s=-2} = -\left(\frac{26(-2)^3 + 75(-2)^2 + 72(-2) + 26}{(-2 + 1)^2}\right) = 26 \]
Calculate Second Derivative

- The second derivative of $Y(s)$ must be calculated to find $C_1$

$$Y'(s) = -\frac{26s^3 + 75s^2 + 72s + 26}{(s + 1)^2} = -(26s^3 + 75s^2 + 72s + 26)(s + 1)^{-2}$$

$$Y''(s) = -(26s^3 + 75s^2 + 72s + 26)(-2)(s + 1)^{-3} - (s + 1)^{-2}(78s^2 + 150s + 72)$$

$$= -\left(\frac{78s^2 + 150s + 72}{(s + 1)^2}\right) + 2\left(\frac{26s^3 + 75s^2 + 72s + 26}{(s + 1)^3}\right)$$

$$C_1 = \left.\frac{Y''(s)}{2}\right|_{s=-2}$$

$$= \frac{1}{2} \left[-\left(\frac{78s^2 + 150s + 72}{(s + 1)^2}\right) + 2\left(\frac{26s^3 + 75s^2 + 72s + 26}{(s + 1)^3}\right)\right]_{s=-2}$$

$$= -16$$
Result of Expansion

• The Rational Part of $F(s)$ is expanded to:

$$F_o(s) = \frac{-13s^3 - 36s^2 - 42s - 16}{(s + 2)^3(s + 1)} = \frac{c_1}{s + 2} + \frac{c_2}{(s + 2)^2} + \frac{c_3}{(s + 2)^3} + \frac{c_4}{s + 1}$$

$$= \frac{-16}{s + 2} + \frac{26}{(s + 2)^2} + \frac{-28}{(s + 2)^3} + \frac{3}{s + 1}$$

• Thus $F(s)$ can be rewritten as:

$$F(s) = 2 + F_0(s) = 2 + \frac{3}{s + 1} - \frac{28}{(s + 2)^3} + \frac{26}{(s + 2)^2} - \frac{16}{s + 2}$$
Table Method Solution

- The following **transform pairs** can be used to evaluate $f(t)$

  - **The Inverse Laplace Transform** can be calculated directly

$$F(s) = 2 + \frac{3}{s+1} - \frac{28}{(s+2)^3} + \frac{26}{(s+2)^2} - \frac{16}{s+2}$$

$$f(t) = 2\delta(t) + \left\{ 3e^{-t} - 14t^2e^{-2t} + 26te^{-2t} - 16e^{-2t} \right\}u(t)$$
**Example**

- Find the inverse Laplace transform of

\[
F(s) = \frac{10(s + 3)}{(s + 1)^3(s + 2)}
\]

\[
F(s) = \frac{10(s + 3)}{(s + 1)^3(s + 2)} = \frac{K_{11}}{s + 1} + \frac{K_{12}}{(s + 1)^2} + \frac{K_{13}}{(s + 1)^3} + \frac{K_2}{s + 2}
\]

\[
K_{13} = (s + 1)^3 F(s) \bigg|_{s=-1} = \frac{10(2)}{(1)} = 20
\]

\[
K_{12} = \frac{d}{ds} \left( (s + 1)^3 F(s) \right) \bigg|_{s=-1} = \frac{d}{ds} \left( \frac{10(s + 3)}{s + 2} \right) \bigg|_{s=-1} = -10
\]

\[
K_{11} = \frac{1}{2!} \frac{d^2}{ds^2} \left( (s + 1)^3 F(s) \right) \bigg|_{s=-1} = \frac{1}{2!} \frac{d}{ds} \frac{-10}{(s + 2)^2} = 10
\]

\[
K_2 = (s + 2) F(s) \bigg|_{s=-2} = \frac{10(1)}{(-1)^3} = -10
\]

\[
f(t) = \left( 10e^{-t} - 10te^{-t} + 20 \left( \frac{1}{2} t^2 e^{-t} \right) - 10e^{-2t} \right) u(t)
\]
Special Case: Repeated Complex Poles

• Find the inverse Laplace of

\[ F(s) = \frac{768}{(s^2 + 6s + 25)^2} \]

\[ F(s) = \frac{K_1}{(s + 3 - j4)^2} + \frac{K_2}{s + 3 - j4} + \frac{K_1^*}{(s + 3 + j4)^2} + \frac{K_2^*}{s + 3 + j4} \]

\[ K_1 = \frac{768}{(s + 3 + j4)^2} \bigg|_{s=-3+j4} = \frac{768}{(j8)^2} = -12. \]

\[ K_2 = \frac{d}{ds} \left[ \frac{768}{(s + 3 + j4)^2} \right]_{s=-3+j4} = -\frac{2(768)}{(s + 3 + j4)^3} \bigg|_{s=-3+j4} = -j3 = 3 \angle -90^\circ. \]

\[ f(t) = [-24te^{-3t} \cos(4t) + 6e^{-3t} \cos(4t - 90^\circ)]u(t). \]
Summary

- Direct calculation of Inverse Laplace Transform is difficult
  - Practically, the Inverse Laplace Transform of a rational function is calculated using a table look-up method
  - Use long division and partial fraction expansion to put $F(s)$ in strictly rational form
  - Two general types of poles: distinct (including complex) and repeated (including complex).

Read the posted file from UPenn on how to do Laplace or inverse Laplace Transform in Matlab